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Locally homogeneous pseudo-Riemannian manifolds

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Abstract

The work of Cartan, Nomizu, Singer, Tricerri and Vanhecke on manifolds with transitive algebras of Killing vector fields is extended to the pseudo-Riemannian case.

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0. Introduction and Summary

The homogeneous universes in space and time "display in simple form features of more complex expanding universes" [19, Ch. 7]. Some of these space–times remain a starting point for studies in general relativity or electromagnetism [3,11,13,20]. Equally important in cosmology is the local metric classification of three dimensional geometries viewed as slices in homogeneous spacetimes [8]. Wesson theory [24] and supergravity [17] are other reasons for a geometric study of locally homogeneous pseudo-Riemannian manifolds (l.h.pR.'s). Ref. [7] gives a recent account of the local theory of homogeneous pseudo-Riemannian structures.

Our approach to the study of l.h.pR.'s in their full generality is different, and uses as a starting point the theory of Cartan triples [15]. Since in some respects the extension of that method is obvious, proofs will be kept to a minimum.

Suppose the connected pseudo-Riemannian manifold of index ν, (M, g) enjoys the following property: there is a Lie algebra $\mathfrak{k}$ of Killing vector fields on M, so that each tangent vector of M extends to an element of $\mathfrak{k}$. Such a pseudo-Riemannian manifold is said *locally homogeneous* (l.h.pR.) and $\mathfrak{k}$ is a *transitive Killing algebra* on M.

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Let K be the abstract group associated with $\mathfrak{k}$ and let u be an orthoframe at a given point x on M. Then one may exponentiate the infinitesimal action of a neighborhood of the identity in K into the orthoframe bundle OM (local version of $IM \subseteq OM$). The tangent map defines a monomorphism of $\mathfrak{k}$ to the tangent space at u to OM. The structure equations of $\mathfrak{k}$ are obtained by pulling back the structure equations of the Ambrose–Singer connection and tautological one-form on the reduced bundle. This gives a decomposition of $\mathfrak{k}$ into $\mathfrak{g}_u \oplus \mathbb{R}^n$, where $\mathfrak{g}_u$ (the algebra of the structure group of the reduced bundle) is the u-isotropic representation of the isotropy algebra into the pseudo-orthogonal algebra $\mathfrak{o}_\nu(n)$. The bracket on $\mathfrak{g}_u \oplus \mathbb{R}^n$ is a modification of the standard semiproduct with extra terms arising from two operators. One is the *Cartan–Singer map* $\Gamma_u : \mathbb{R}^n \to \mathfrak{p}$, where $\mathfrak{p}$ is a complement of $\mathfrak{g}_u$ in $\mathfrak{o}_\nu(n)$ (the orthocomplement w.r.t. the Killing form, whenever this exists); it is defined by pulling back the original $\mathfrak{o}_\nu(n)$-connection restricted to $\mathfrak{g}_u$. The other operator in the definition of the bracket is the curvature at x w.r.t. u of the Ambrose–Singer connection, and is determined by Γ_u and by Ω_u, the $\mathfrak{g}_u$-projection of the curvature along $\mathfrak{p}$; $(\mathfrak{p}, \Gamma_u, \Omega_u)$ is said to be a $\mathfrak{g}_u$-*triple*.

The l.h.pR. (M, g) is locally isometric to a homogeneous space, iff the connected subgroup of Lie algebra $\mathfrak{g}_u$ of the abstract Lie group K of Lie algebra $\mathfrak{g}_u \oplus \mathbb{R}^n$ is closed in K. An example of l.h.p.R. which is locally nonisometric to a homogeneous space (see also Refs. [9,10,15,16]) is displayed in Section 3.

The converse is also true: if $\mathfrak{g}$ is a subalgebra of $\mathfrak{o}_\nu(n)$, and $(\mathfrak{p}, \Gamma, \bar{\Omega})$ is a $\mathfrak{g}$-triple, there is a l.h.pR. (M, g), unique up to a local isometry, called the *local geometric realization* of the $\mathfrak{g}$-triple, and a frame $u \in OM$, for which $\mathfrak{g}$ is the linear isotropy algebra w.r.t. u; moreover $\Gamma_u = \Gamma$ and $\Omega_u = \bar{\Omega}$. As such, the problem of listing the n-dimensional l.h.pR.'s of index ν amounts to the following algorithm:

(a) find conjugacy classes of Lie subalgebras of $\mathfrak{o}_\nu(n)$;

(b) for a given Lie subalgebra $\mathfrak{g}$ of $\mathfrak{o}_\nu(n)$, find all $\mathfrak{g}$-triples.

This method is not too effective if $\mathfrak{g} = 0$ (pseudo-Riemannian Lie groups). However, starting from the joint work of Cordero and Parker [6] and using Propositions 3.2 and 3.3 in this study, the program can be carried out completely in dimension three, and even in this low-dimensional case there are examples of l.h.pR.'s that are *degenerated* (see Section 2 for a definition) or of nonsymmetric Lorentz manifolds, modelled on a symmetric space [5,16]. The + and − spaces which are introduced in Section 3 are typical examples of nonflat Lorentz manifolds with null nongeneric vectors [1].

1. Transitive Killing algebras of pseudo-Riemannian manifolds

Assume g is a pseudo-Riemannian structure of index ν on the n-dimensional simply connected manifold M and that $\mathfrak{k}$ is a transitive Killing algebra on (M, g). The kernel of the evaluation map $\mathrm{ev}_x : \mathfrak{k} \to T_xM$ is the *isotropy subalgebra* $\mathfrak{k}_x$ of $\mathfrak{k}$ at the point x.

If $\xi \in \mathfrak{k}_x$, the local one-parameter group of isometries generated by ξ, (φ_t^ξ), has the fixed point x; consequently, for each u in OM_x, one has a local one parameter subgroup $\Lambda_\xi(t)$ of the pseudo-orthogonal group $O_\nu(n)$ [12], defined as follows:

Let f: $U \to M$ be a local isometry defined on an open subset U of M and let Lf: $OU \to OM$, be the left of f to the bundle of orthoframes. Then

$$(L\varphi_t^\xi)(u) = u \cdot \Lambda_\xi(t) .$$

The *linear isotropy representation* of $\mathfrak{k}_x$ associated with the frame u is λ_u: $\mathfrak{k}_x \to \mathfrak{o}_\nu(n)$,

$$\lambda_u(\xi) = \dot{\Lambda}_\xi(0) . \tag{1.1}$$

Note that the main difference between the Riemannian and the other $O_\nu(n)$-structures is that $\mathfrak{o}(n)$ is the only compact form among the real forms $\mathfrak{o}_\alpha(n)$ of $\mathfrak{o}(n, \mathbb{C})$. Therefore, the method of Cartan triples [15] can be restated in the pseudo-Riemannian case whenever the restriction of the Killing form to $\mathfrak{g}_u = \lambda_u(\mathfrak{k}_x)$ is *nondegenerate*. A l.h.pR. is *nondegenerate* (n.l.h.pR.) if it admits at least one transitive Killing algebra $\mathfrak{k}$ with a nondegenerate linear isotropy algebra $\mathfrak{g}_u$. Such a $\mathfrak{k}$ is said to be a *nonsingular Killing algebra*.

Let (K, H) be the pair consisting of the simply connected group of Lie algebra $\mathfrak{k}$, and of its connected Lie subgroup of Lie algebra $\mathfrak{k}_x$, and let α be the maximal local K-transformation group on M [14] generated by $\mathfrak{k}$.

The map α lifts in a standard way to a local K-transformation group of isometries without fixed points $L(\alpha)$ of (OM, g_∇), where g_∇ is the metric associated to the Levi-Civita connection, defined on the basic and fundamental vector fields in the following formulas [22]:

$$g_\nabla(B_u(X), B_u(Y)) = \langle X, Y \rangle_\nu , \quad X, Y \in \mathbb{R}^n ,$$

$$g_\nabla(A_u^*, B_u^*) = -\operatorname{Tr} AB , \quad A, B \in \mathfrak{o}_\nu(n) ,$$

$$g_\nabla(B_u(X) \cdot A_u^*) = 0 , \quad X \in \mathbb{R}^n, A \in \mathfrak{o}_\nu(n) , \tag{1.2}$$

where $\langle \ , \ \rangle_\nu$ is the standard pseudo-Euclidean scalar product of index ν.

Let D be an open neighborhood of O in $\mathfrak{k}$, such that $\varphi_1^\xi(x)$ is defined for each $\xi \in D$. If $u \in OM_x$, one may define the map J_u: $\exp D \to OM$, by

$$J_u(\exp \xi) = L(\alpha)(\exp \xi, u) . \tag{1.3}$$

Then, if $\tilde{\xi}$ is the Levi-Civita horizontal lift of ξ, and if $A_\xi = L_\xi - \nabla_\xi$, we obtain, as in the Riemannian case:

Proposition 1.1. *Let $[A_{\xi x}]_u$ be the matrix of $A_{\xi x}$ w.r.t. u. Then*

$$(d_1 J_u)(\xi) = \tilde{\xi}(u) - ([A_{\xi x}]_u)_u^* . \tag{1.4}$$

Let $\mathfrak{p}$ be a complement of $\mathfrak{g}_u$ in $\mathfrak{o}_\nu(n)$. From the previous proposition, it follows that $d_1 J_u$ is one to one, so that if H is the horizontal Levi-Civita distribution, and if

σ_u: $\mathfrak{o}_\nu(n) \to T_u OM$ is the map $A \mapsto A^*_u$, "tangent" to the right action of $O_\nu(n)$ in OM_u, then

$$\mathfrak{m}_u = (d_1 J_u)^{-1}(\sigma_u(\mathfrak{p}) + H_u) \tag{1.5}$$

is a direct summand of $\mathfrak{k}_x$ in $\mathfrak{k}$.

As such, the restriction of ev_u to $\mathfrak{m}_u$ is a linear isomorphism from $\mathfrak{m}_u$ to $T_x M$. Then, if $u = (x, u_1, \ldots, u_n)$, for each index $i = 1, \ldots, n$, there is a unique ξ_i in $\mathfrak{m}_u$, such that $\xi_i(x) = u_i$. One may prove the following:

Proposition 1.2. *Let $\theta \in \mathscr{D}^1(OM, \mathbb{R}^n)$, $\omega \in \mathscr{D}^1(OM, \mathfrak{o}_\nu(n))$ be the tautological form and the Levi-Civita connection form on* $\exp D$, and let ${}_u\theta = J^*_u \theta$, ${}_u\omega = J^*_u \omega$. Then ${}_u\theta$ *and* ${}_u\omega$ *are left invariant forms on* $\exp D$ *and* $\operatorname{rank} {}_u\theta = n$.

Further, ${}_u\omega$ splits into two vector-valued parts, ${}_u\omega = {}_u\omega_\mathfrak{p} \oplus {}_u\omega_\mathfrak{g}$. Let $\|X\|^2_\nu = \langle X, X\rangle_\nu$. Then Cartan's theorem on the local structure of a homogeneous Riemannian space [4, Ch. XII] has the following analogue:

Theorem 1.1. *(1) There is a linear map Γ_u: $\mathbb{R}^n \to \mathfrak{p}_u$, such that ${}_u\omega_\mathfrak{p} = \Gamma_u \circ {}_u\theta$.*

(2) There is a neighborhood V of 1_K, which is regular for the foliation F, given by the system ${}_u\theta = 0$. F is a pseudo-Riemannian foliation with the transverse metric $\|{}_u\theta\|^2_\nu$ which induces a locally K-invariant metric g_u on the space of leaves V/F.

(3) Let F_k be the leaf of F through k. The map $F_k \to k(x)$ is a local isometry between $(V/F, g_u)$ and (M, g).

We shall say that Γ_u is the *Cartan–Singer map w.r.t. the decomposition* $\mathfrak{o}_\nu(n) = g_u \oplus \mathfrak{p}$. Let us look for the Maurer–Cartan equations of $\mathfrak{k}$ as a consequence of the structure equations of OM.

First, let $\Omega \in \mathfrak{D}^2(OM, \mathfrak{o}_\nu(n))$ be the Riemann curvature form, and let ${}_u\Omega = J^*_u \Omega$. ${}_u\Omega$ splits into its $\mathfrak{g}_u$ and $\mathfrak{p}$ components:

$${}_u\Omega = {}_u\Omega_\mathfrak{g} \oplus {}_u\Omega_\mathfrak{p} . \tag{1.6}$$

Let (ϵ_b), $b = 1, \ldots, \frac{1}{2}n(n-1)$, be a basis of $\mathfrak{o}_\nu(n)$, such that the first elements lie in $\mathfrak{g}_u$ and the last ones in $\mathfrak{p}$; if α is the index for the elements in $\mathfrak{g}_u$, let ${}_u\Omega_\mathfrak{g}$ be the vector-valued form $\frac{1}{2}{}_u\Omega^\alpha_{ij}\,{}_u\theta^i \wedge {}_u\theta^j \epsilon_\alpha$.

Since ${}_u\Omega^\alpha_{ij}$ are constant on $\exp D$, one may define the bilinear skew symmetric map Ω_n: $\mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{g}_u$ by

$$\Omega_u(e_i, e_j) = {}_u\Omega^\alpha_{ij} \epsilon_\alpha, \quad i, j = 1, \ldots, n . \tag{1.7}$$

We call the map Ω_u the *$\mathfrak{g}_u$-curvature of M, w.r.t. the decomposition* $\mathfrak{o}_\nu(n) = \mathfrak{g}_u \oplus \mathfrak{p}$. Let T: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\bar{\Omega}$: $\mathbb{R}^n \times \mathbb{R}^n \to \mathfrak{g}_u$, be defined by

$$T(X, Y) = \Gamma_u(Y)X - \Gamma_u(X)Y , \tag{1.8}$$

$$\bar{\Omega}(X, Y) = \Omega_u(X, Y) - [\Gamma_u(X), \Gamma_u(Y)]_{\mathfrak{g}_u} . \tag{1.9}$$

The bracket in (1.9) is the commutator. The following result is a consequence of the structure equations on OM, pulled back on K, as in Proposition 1.2.

Theorem 1.2. *$\mathfrak{k}$ is isomorphic to the Lie algebra* $(\mathfrak{g}_u \oplus \mathbb{R}^n, [\ ,\ ])$:

$$[\xi, \eta] = [\xi, \eta]\ , \quad \forall \xi, \forall \eta \in \mathfrak{g}_u\ , \tag{1.10}$$

$$[\xi, X] = \xi(X) + [\xi, \Gamma_u(X)]_{\mathfrak{g}_u}, \quad \forall \xi \in \mathfrak{g}\ , \forall X \in \mathbb{R}^n\ , \tag{1.11}$$

$$[X, Y] = -T(X, Y) - \tilde{\Omega}(X, Y)\ , \quad \forall X\ , \forall Y \in \mathbb{R}^n\ . \tag{1.12}$$

Remark 1.1 If the transitive Killing algebra is nonsingular, we shall always take for $\mathfrak{p} = \mathfrak{p}_u$ the orthocomplement of $\mathfrak{g}_u$ in $\mathfrak{o}_\nu(n)$ w.r.t. the Killing form. In this case, $\mathfrak{m}_u = \mathfrak{m}$ does not depend on u, and $\mathfrak{k} = \mathfrak{k}_x \oplus \mathfrak{m}$ is a reductive decomposition. As in the Riemannian case, the *canonical connection* of the n.l.h.pR. M w.r.t. this decomposition has *torsion T*, and *$\mathfrak{g}_u$-part of the curvature $\tilde{\Omega}$*. Ω_u is called the *$\mathfrak{g}_u$-part of the curvature*, and Γ_u the *Cartan–Singer map* since the Ambrose–Singer connection refers to the decomposition $\mathfrak{o}_\nu(n) = \mathfrak{g}_u \oplus \mathfrak{g}_u^\perp$.

As a consequence of Theorem 1.2, the Jacobi identities for $\mathfrak{k}$ are as follows:

$$\begin{aligned}
&[\xi, \tilde{\Omega}(X, Y)] - \tilde{\Omega}(\xi X, Y) - \tilde{\Omega}(X, \xi Y) \\
&\quad + [[\xi, \Gamma_u(X)]_{\mathfrak{g}_u}, \Gamma_u(Y)]_{\mathfrak{g}_u} + [\xi, \Gamma_u(T(X, Y))]_{\mathfrak{g}_u} \\
&\quad + [\Gamma_u(X), [\xi, \Gamma_u(Y)]_{\mathfrak{g}_u}]_{\mathfrak{g}_u} = 0\ , \quad \forall \xi \in \mathfrak{g}\ , \forall X, \forall Y \in \mathbb{R}^n\ ;
\end{aligned} \tag{1.13}$$

$$\begin{aligned}
&\sum_{\text{cycl}} \tilde{\Omega}(T(X, Y), Z) - [\tilde{\Omega}(X, Y), \Gamma_u(Z)]_{\mathfrak{g}_u} = 0\ , \\
&\quad \forall X, \forall Y, \forall Z \in \mathbb{R}^n\ ;
\end{aligned} \tag{1.14}$$

$$\begin{aligned}
&\sum_{\text{cycl}} \tilde{\Omega}(X, Y)(Z) - T(T(X, Y)\ Z) = 0\ , \\
&\quad \forall X, \forall Y, \forall Z \in \mathbb{R}^n\ .
\end{aligned} \tag{1.15}$$

The ad $\mathfrak{g}_u$-invariance of Γ_u, valid in the Riemannian case, becomes:

$$\Gamma_u(\xi X) = [\xi, \Gamma_u(X)]_{\mathfrak{p}}\ , \quad \forall \xi \in \mathfrak{g}_u, \forall X \in \mathbb{R}^n\ . \tag{1.16}$$

The $\mathfrak{p}$-part of the curvature, ${}_{\mathfrak{p}}\Omega_u$, is given by the same formula as in the Riemannian case:

$${}_{\mathfrak{p}}\Omega_u = [\Gamma_u(X), \Gamma_u(Y)]_{\mathfrak{p}} + \Gamma_u(T(X, Y))\ . \tag{1.17}$$

The analogue of Theorem 1.3 in Ref. [15] is:

Theorem 1.3. *Let $\mathfrak{k}$ be a transitive Killing algebra of the l.h.pR. M and let $\mathfrak{h}$ be the isotropy algebra at point x. Then M is locally isometric to a homogeneous pseudo-Riemannian space iff H is closed in K.*

We also have

Proposition 1.3. *Let $\mathfrak{k}$ be a nonsingular Killing algebra of M. Then the sequence of the covariant derivatives of the Riemannian curvature tensor, $(\nabla^s R)_{s\in\mathbb{N}}$, w.r.t. the frame u, may be recovered from the Cartan–Singer map Γ_u and from the $\mathfrak{g}_u$-curvature Ω_u, by means of the formulas*:

$$\hat{\Omega}_u = \Omega_u + {}_{\mathfrak{p}}\Omega_u\,, \tag{1.18}$$

$$(\nabla^0 R)(X,Y;Z,T) = \langle \hat{\Omega}_u(u^{-1}X, u^{-1}Y)u^{-1}T, u^{-1}Z\rangle_\nu\,, \tag{1.19}$$

$$\iota_X \nabla^{s+1} R = \Gamma_u(u^{-1}X)\cdot \nabla^s R\,, \tag{1.20}$$

where ι_X is the interior product and $\Gamma_u(u^{-1}X)$ acts as a derivation.

Remark 1.2. *The Riemannian curvature tensor of M at x w.r.t. u is given by (1.19) even if M is degenerated.*

2. $\mathfrak{g}$-triples

Definition 2.1. Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{o}_\nu(n)$. We say that $(\mathfrak{p}, \Gamma, \bar{\Omega})$ is a $\mathfrak{g}$-*triple* if $\mathfrak{o}_\nu(n) = \mathfrak{g}\oplus\mathfrak{p}$, $\Gamma: \mathbb{R}^n \to \mathfrak{p}$ is a linear map and $\bar{\Omega}$: $\mathbb{R}^n\times\mathbb{R}^n \to \mathfrak{g}$ is a bilinear anti-symmetric map, such that if we formally replace $\mathfrak{g}$ with $\mathfrak{g}_u$, Γ with Γ_u, and $\bar{\Omega}$ with Ω_u, then (1.13)–(1.16) will hold true.

If the restriction of the Killing form to $\mathfrak{g}$ is nondegenerate, we say that the $\mathfrak{g}$-triple $(\mathfrak{g}^\perp, \Gamma, \bar{\Omega})$ is a *Cartan triple.*

Theorem 2.1. *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{o}_\nu(n)$, and let $(\mathfrak{p}, \Gamma, \bar{\Omega})$ be a $\mathfrak{g}$-triple. Then there is a l.h.pR (M, g) unique up to a local isometry, a frame $u \in OM$, and a transitive Killing algebra $\mathfrak{k}$ on M, such that $\lambda_u(\mathfrak{k}_{\pi(u)}) = \mathfrak{g}$, and w.r.t. the decomposition $\mathfrak{o}_\nu(n) = \mathfrak{g}\oplus\mathfrak{p}$, $\Gamma_u = \Gamma$* and *$\Omega_u = \bar{\Omega}$.*

We shall say that (M, g) in Theorem 2.1 is the *local geometric realization* of the $\mathfrak{g}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$.

Proof. We consider the Lie algebra $\mathfrak{k} = (\mathfrak{g}\oplus\mathbb{R}^n, [\ ,\ ])$, where $[\ ,\ ]$ is defined in (1.10)–(1.12), with $(\mathfrak{g}, \mathfrak{p}, \Gamma, \bar{\Omega})$ in place of $(\mathfrak{g}_u, \mathfrak{p}, \Gamma_u, \Omega_n)$. It is obvious that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{k}$. Let G be the connected subgroup with Lie algebra $\mathfrak{g}$ of the simply connected group K, of Lie algebra $\mathfrak{k}$. Let $\theta\oplus\omega \in \mathscr{D}^1(K, \mathbb{R}^n\oplus\mathfrak{g})$ be the canonical form of K, and let $V = \exp\Delta\cdot\exp U$, with Δ, U open neighborhoods of the zeros of $\mathbb{R}^n$ and $\mathfrak{g}$, and Δ such that through each point of $\exp\Delta$ there passes a unique leaf of the foliation F defined on V by $\theta = 0$.

As $\|\theta\|_\nu^2$ is constant along the leaves of F, this tensor field on V is projectable to a pseudo-Riemannian metric g on $M = V/F$.

The tangent space at x, the leaf of 1_K, is isomorphic to the quotient $\mathfrak{k}/\mathfrak{g}$. We shall consider then the frame u of components $u_i = e_i + \mathfrak{g}$, which belongs to OM_x.

Let α be the natural local K transformation group of M, defined on some open neighborhood of $1_K \times M$ in $K \times M$, induced by the left translation of K. We claim that α is almost effective. Suppose it is not. Then there exists a nonzero $\xi \in \mathfrak{g}$ and a sequence $t_n \neq 0$, converging to zero, such that $\alpha_{\exp t_n \xi}$ is the identity of some neighborhood of the leaf of 1_K in M.

It follows that there exists a neighborhood N of $0 \in \mathbb{R}^n$, such that for each $X \in N$, there is some $\eta_n \in \mathfrak{g}$, such that $\exp t_n \xi \cdot \exp X = \exp X \cdot \exp \eta_n$. This condition expresses the fact that we remain on the same leaf of F, as we act by $\exp t_n \xi = \mathrm{id}$. Then due to a consequence of the Campbell–Hausdorff formula [21, Th. 5.16], if we change X in the above formula to $t_n X$, we deduce that $t_n^2[\xi, X] + o(t_n^3) = -t_n \xi + \eta_n \in \mathfrak{g}$.

Then $[\xi, X]$ is in $\mathfrak{g}$, as a limit of elements in $\mathfrak{g}$, and $\mathrm{ad}\, \xi(\mathbb{R}^n) \subseteq \mathfrak{g}$. From (1.11), this is possible iff $\xi = 0$, thereby proving our claim and showing that $\lambda_u(\mathfrak{k}_x) = \mathfrak{g}$.

Further, since the Lie algebras $\mathfrak{k}$ and $\mathfrak{k}_u$ associated to the $\mathfrak{g}$-triples $(\mathfrak{p}, \Gamma, \bar{\Omega})$ and $(\mathfrak{p}_u, \Gamma_u, \Omega_u)$ have the same structure equations, $\Gamma = \Gamma_u$ and $\bar{\Omega} = \Omega_u$.

Let K be the simply connected Lie group of Lie algebra $\mathfrak{k} = (\mathfrak{g} \oplus \mathbb{R}^n, [\ ,\ ])$, associated to the $\mathfrak{g}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$, and let G be the connected subgroup of K, of Lie algebra $\mathfrak{g}$. If G is closed in K, then, as in the proof of Theorem 1.4., $\|\theta\|_\nu^2$ is projectable to a pseudo-Riemannian metric g on K/G. We shall say that $(K/G, g)$ is the *geometric realization of the closed* $\mathfrak{g}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$.

The geometric realization of a $\mathfrak{g}$-triple is simply connected; moreover, if $\mathfrak{k}$ has an n-dimensional subalgebra $\mathfrak{l}$, which is transverse to $\mathfrak{g}$, then the geometric realization is diffeomorphic to the simply connected group of Lie algebra $\mathfrak{l}$.

Until now, there were no relevant differences between the Riemannian and the pseudo-Riemannian case. However, if we try to generalize the equivalence criterion of section 3 in Ref. [15], we encounter some difficulties even for n.l.h.pR.'s, since the nondegeneracy of a transitive Killing algebra may not be inherited from the whole Lie algebra of Killing vector fields on M, $\mathfrak{k}(M)$. All we can prove is the following:

Theorem 2.2. *Let M_1, M_2 be two l.h.pR.'s. Then there is a local isometry f from M_1 to M_2, iff there are some frames $u_1 \in OM_1$, $u_2 \in OM_2$, such that $\lambda_{u_1}(\mathfrak{k}(M_1)_{\pi u_1}) = \lambda_{u_2}(\mathfrak{k}(M_2)_{\pi u_2}) = \mathfrak{g}$ and there is a complement $\mathfrak{p}$ of $\mathfrak{g}$ in $\mathfrak{o}_\nu(n)$, so that w.r.t. the decomposition $\mathfrak{o}_\nu(n) = \mathfrak{g} \oplus \mathfrak{p}$, the Cartan–Singer maps and the $\mathfrak{g}$-parts of the curvature of M_1 and M_2 are equal.*

For $\alpha = 1, 2$, let $\mathfrak{k}_\alpha$ be a transitive Killing algebra of some n-dimensional l.h.pR. M_α of index ν, and let $u_\alpha \in OM_\alpha$.

Corollary 2.1. *If M_1 is locally isometric to M_2, their curvature tensors R_{1,u_1}, R_{2,u_2}, given by (1.19), are conjugate under the natural action of $O_\nu(n)$ on the space of curvature tensors.*

If $\mathfrak{k}$ is a transitive Killing algebra of M, $u \in OM_x$ and $\hat{\Omega}_u$ is defined in (1.18), the *Ricci form associated with* $(\mathfrak{k}, u)$ is the bilinear symmetric form ${}_u\rho$: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, given by:

$$ {}_u\rho(e_i, e_j) = \mathrm{Tr}(x \to \hat{\Omega}_u(x, e_i)e_j) . \tag{2.1} $$

The *Ricci polynomial* Ric is defined by:

$$ \mathrm{Ric}(t) = \det({}_u\rho(e_i, e_j) - t\,{}_\nu\delta_{ij}) , \tag{2.2} $$

where ${}_\nu\delta_{ij} = \delta_{ij}$ for $i \leqslant n-\nu$ and ${}_\nu\delta_{ij} = -\delta_{ij}$ for $i > n-\nu$.

Remark 2.1. $\mathrm{Ric}(t)$ is an invariant of the local isometry class of the l.h.pR. (M, g).

Remark 2.2. If $\mathfrak{k}$ is a nonsingular Killing algebra of (M, g), and if $\mathfrak{g} = \lambda_u(\mathfrak{k}_x)$, then, as in the Riemannian case, one may find $\lambda_u(\mathfrak{k}(M)_x)$, starting from the Cartan triple $(\mathfrak{g}^\perp, \Gamma_u, \bar{\Omega}_u)$, as follows:

$$ \lambda_u(\mathfrak{k}(M)_x) = \{\xi \in \mathfrak{o}_\nu(n),\ \xi \cdot \nabla^s R = 0,\ \forall s \in \mathbb{N}\} . $$

In this case, one may label as $\mathfrak{g}_s$ the vector subspace $\{\xi \in \mathfrak{o}_\nu(n),\ \xi \cdot \nabla^p R = 0,\ p \leqslant s\}$ of $\mathfrak{o}_\nu(n)$, and define the *Singer invariant* to be the largest s, such that $\mathfrak{g}_s \neq \mathfrak{g}_\infty$.

3. Examples

This section provides applications of the mechanics of $\mathfrak{g}$-triples. A first example proves the consistency of Theorem 1.3.

In order to obtain examples relevant to that theorem, it is natural to look for a transitive Killing algebra, whose linear isotropy subalgebra is the Lie algebra of a nonclosed Lie subgroup of $O_\nu(n)$.

As a vector space, $\mathfrak{o}_\nu(n)$ has the basis $(f_i^j)_{1 \leqslant i < j \leqslant n}$,

$$ f_i^j = E_i^j - {}_\nu\delta_{ij}\, E_j^i . \tag{3.1} $$

In our example $n=5$ and $\nu=3$, and we start from the maximal toral subalgebra $\mathfrak{t} = \mathbb{R}f_1^2 \oplus \mathbb{R}f_3^4$ of $\mathfrak{o}_3(5)$, tangent to the torus T.

Let r be a positive irrational number. Then, the one-parameter subgroup G_r of the Lie algebra $\mathfrak{g}_r = \mathbb{R}(f_1^2 + rf_3^4)$ is dense in T, and therefore we will look for a Cartan triple $(\mathfrak{g}, \Gamma, \bar{\Omega})$ with $\mathfrak{g} = \mathfrak{g}_r$. Let $f = f_1^2 + rf_3^4$.

One of the solutions for (1.16)–(1.18) is

$$\Gamma(e_1)=bf_2^5,\qquad \Gamma(e_2)=-bf_1^5,\qquad \Gamma(e_3)=df_4^5,$$
$$\Gamma(e_4)=-df_3^5,\qquad \Gamma(e_5)=0,$$
$$\bar{\Omega}(e_1,e_2)=b\left(\frac{b}{1+r^2}+\frac{2d}{r}\right)f,$$
$$\bar{\Omega}(e_3,e_4)=-d\left(\frac{rd}{1+r^2}+2b\right)f,\quad b>0, d>0, br-d<0. \tag{3.2}$$

The corresponding transitive Killing algebra $\mathfrak{k}=(\mathfrak{g}\oplus\mathbb{R}^5, [\ ,\ ])$ splits as a direct sum of two copies of $\mathfrak{su}(2)$, $\mathfrak{k}=\mathfrak{k}_1\oplus\mathfrak{k}_2$, where $\mathfrak{k}_1=\mathrm{Span}(e_1, e_2, re_5-df)$, $\mathfrak{k}_2=\mathrm{Span}(e_3, e_4, -e_5+bf)$.

The subalgebra $\mathfrak{s}=\mathbb{R}(-e_5+bf)\oplus\mathbb{R}(re_5-df)$ is toral in $\mathfrak{k}$. The Lie subgroup of H of the Lie algebra $\mathfrak{g}$, of the simply connected group $K=\mathrm{SU}(2)\times\mathrm{SU}(2)$ of the Lie algebra $\mathfrak{k}$, is dense in the torus S of the Lie algebra $\mathfrak{s}$.

Theorem 3.1. *For any positive irrational r, the local geometric realization of the Cartan triple* $(\mathfrak{g}_r, \Gamma, \bar{\Omega})$*, defined in (3.2), is a five-dimensional n.l.h.pR. of index 3, that is not locally isometric to a homogeneous space.*

Remark 3.1. (See Ref. [23] for a Riemannian analogue.) A local geometric realization of the Cartan triple (3.2) may be viewed as a transverse manifold M to the pseudo-Riemannian foliation D_r of $SU(2)\times SU(2)$, with the left invariant metric g defined by the ad $\mathfrak{h}$-invariant bilinear form B on $\mathfrak{su}(2)$, $B((x_1,y_1),(x_2,y_2))=-\lambda\,\mathrm{Tr}(x_1,y_1)+\mu\,\mathrm{Tr}(x_2,y_2)$,

$$\lambda=\frac{r}{b(d-br)},\qquad \mu=\frac{1}{d(d-br)}.$$

D_r is the Lie foliation generated by the diagonal matrices

$$\begin{pmatrix} e^{it} & & & \\ & e^{-it} & & \\ & & e^{irt} & \\ & & & e^{-irt}\end{pmatrix}$$

and the factor metric g.

Our next objective is to find the three-dimensional degenerate l.h.pR.'s. Let us denote $f_1^2+f_1^3$ by f. In order to exhaust all the possibilities, we need the following elementary fact:

Lemma 3.1. *A degenerate Lie subalgebra of* $\mathfrak{o}_1(3)$ *is conjugate to* $\mathfrak{m}(3)=\mathrm{Span}(f, f_2^3)$ *or to* $\mathfrak{h}t=\mathbb{R}f$.

It may be shown that the local geometric realization of an $\mathfrak{m}(3)$-triple has constant negative curvature.

If one identifies $\mathfrak{o}_1(3)$ with the Lie algebra of the full group of isometries of the hyperbolic plane, $\mathfrak{ht}$ is the Lie algebra of the horocyclic translations [6, p. 3].

We list the $\mathfrak{ht}$-triples. Since the Killing form vanishes along $\mathfrak{ht}$, we shall take for its complement in $\mathfrak{o}_1(3)$, the plane $\mathfrak{p}=\mathrm{Span}(f\,^3_1, f\,^3_2)$.

Any $\mathfrak{ht}$-triple of the form $(\mathfrak{p}, \Gamma, \bar{\Omega})$ is given by

$$\Gamma(e_1)=af\,^3_2, \qquad \Gamma(e_2)=\Gamma(e_3)=-af\,^3_1+bf\,^3_2, \qquad \bar{\Omega}(e_i, e_j)=\Omega_{ij}f, \tag{3.3}$$

where

$$\Omega_{12}-\Omega_{13}+a^2=0, \qquad \Omega_{23}=ab=0. \tag{3.4}$$

From (1.9) it follows that:

$$\tilde{\Omega}(e_1, e_2)=\tilde{\Omega}_{12}f, \qquad \tilde{\Omega}(e_1, e_3)=\tilde{\Omega}_{13}f, \qquad \tilde{\Omega}(e_2, e_3)=\Omega_{23}f. \tag{3.5}$$

If $a\neq 0$, the transitive Killing algebra associated to the $\mathfrak{ht}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$, $\mathfrak{k}=\mathfrak{ht}\oplus\mathrm{Span}(e_1, e_2, e_3)$, has the structure equations

$$\begin{aligned} &[e_1, e_2]=2ae_3+(a^2-\Omega_{12})f, \qquad [e_1, e_3]=a(e_2+a_3)-\Omega_{12}f, \\ &[e_2, e_3]=-ae_1, \qquad [f, e_1]=e_3-e_2+af, \qquad [f, e_2]=[f, e_3]=e_1. \end{aligned} \tag{3.6}$$

By (1.17) and (1.18) the curvature of the local geometric realization is

$$\begin{aligned} &\hat{\Omega}(e_1, e_2)=\Omega_{12}f^2_1+(\Omega_{12}+a^2)f\,^3_1, \qquad \hat{\Omega}(e_2, e_3)=a^2f\,^3_2, \\ &\hat{\Omega}(e_1, e_3)=(\Omega_{12}+a^2)f\,^2_1+(\Omega_{12}+2a^2)f\,^3_1, \end{aligned} \tag{3.7}$$

with the Ricci polynomial

$$\mathrm{Ric}(t)=(t+2a^2)^3. \tag{3.8}$$

Proposition 3.1. *There exist a locally homogeneous Lorentz manifold (l.h.L.) which has the Ricci polynomial of a space of constant negative curvature, but is not of constant negative curvature.*

Proof. This is a three-dimensional l.h.pR. which apparently is not dependent only on the Ricci polynomial.

From (3.7), it follows that the local geometric realization M of a $\mathfrak{ht}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$ with $a\neq 0$ has the possible nonzero components of the curvature tensor given by

$$R_{1212}=\Omega_{12}, \qquad R_{1213}=\Omega_{12}+a^2, \qquad R_{1313}=\Omega_{12}+2a^2, \qquad R_{2323}=a^2.$$

Let $[x_1, x_2, x_3]$ be the dual (plückerian) coordinates of some nondegenerate tangent plane $\pi \in G_2(T_xM)$, w.r.t. u, the orthoframe given by Theorem 2.1. The sectional curvature of π is (see Ref. [2]):

$$K_x[x_1, x_2, x_3] = -a^2 + (\Omega_{12} + a^2)\,\frac{(x_2 - x_3)^2}{x_3^2 - x_1^2 - x_2^2}\,, \tag{3.9}$$

showing that the space has constant curvature iff $\Omega_{12} + a^2 = 0$.

Thus, K_x is a rational function defined on the complement of the oval $x_3^2 - x_1^2 - x_2^2 = 0$ (the *null locus*) in $\mathbb{P}^2\mathbb{R}$, whose image is the union of two connected subsets I_t, I_s, of $\mathbb{R}$, corresponding to the timelike and the spacelike planes, respectively. The conic $Q = 0$ is the *homaloidal conic* of the point x. Of course, I_t and I_s are invariant under local isometries, and therefore they are local invariants of a l.h.L.

If $\Omega_{12} + a^2 > 0$, $I_t = [-a^2, \infty)$, and if $\Omega_{12} + a^2 < 0$ then, $I_t = (-\infty, -a^2]$, which proves that there are at least three pairwise nonisometric l.h.L.'s, having the Ricci polynomial of a space of constant negative curvature.

If $a = 0$, $\mathfrak{k} = \mathfrak{ht} \oplus \mathrm{Span}(e_1, e_2, e_3)$ has the structure equations

$$[e_1, e_2] = [e_1, e_3] = -\Omega_{12} f, \qquad [e_2, e_3] = -b(e_3 - e_2)\,,$$
$$[f, e_1] = e_3 - e_2\,, \qquad [f, e_2] = [f, e_3] = e_1 + bf. \tag{3.10}$$

For $b = 0$, $\mathfrak{k} = \mathfrak{ht} \oplus \mathrm{Span}(e_1, e_2, e_3)$ is a reductive decomposition, such that the geometric realization is a Lorentz symmetric space that is indecomposable and does not have constant curvature if $\Omega_{12} \neq 0$; this is immediate, since the Ricci polynomial is t^3 and the curvature is not zero.

Let K be the simply connected group of Lie algebra $\mathfrak{k}$. The connected Lie subgroup of the Lie algebra $\mathfrak{ht}$ is closed in K. We shall now that that there are precisely three distinct geometric realizations of this type.

Indeed, the homoloidal conic is the double line $(x_2 - x_3)^2 = 0$, tangent to the null locus, and the sectional curvature is given by (3.9), where $a = 0$. Then, if $\omega\beta < 0$, and M, M' are two local geometric realizations that are associated with the parameters $\Omega_{12} = \omega$ and $\Omega_{12} = \beta$, respectively, one may assume, w.l.o.g., that $I_t(M) = [0, \infty)$ and $I_t(M') = (-\infty, 0]$, and therefore M and M' are not locally isometric. We shall say that the geometric realization of an $\mathfrak{ht}$-triple with $a = 0$ is a + *space*, if $\Omega_{12} > 0$, and is a − *space* is $\Omega_{12} < 0$.

It is known that a Riemannian manifold modelled on an irreducible symmetric space is locally symmetric [5]. The Lorentzian analogue of this statement fails to be true:

Proposition 3.2. *Let α be a real root of $\alpha^2 - \alpha b + \Omega_{12} = 0$. Let $Sol(b, \alpha)$ be the Lie group of affinities of the real plane, generated by the translations $x' = x + u$, $y' = y + v$, and by the dilations $x' = \exp(t)x$, $y' = \exp(bt/\alpha)y$, together with the Lorentz metric*

$$g=\exp(2t)\cdot du^2+\frac{1}{\alpha^2}(2\exp(bt/\alpha)\,dt\cdot dv-dt^2)\,. \qquad (3.11)$$

If $b\Omega_{12}\neq 0$, then $Sol(\mathrm{b},\alpha)$ is modelled on a symmetric space, without being locally symmetric.

Proof. If $\Omega_{12}\neq 0$, the geometric realization of the $\mathfrak{h}\mathfrak{t}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$, defined by (3.3)–(3.5), with $a=0\neq\Omega_{12}$, has $\mathfrak{k}=\mathfrak{h}\mathfrak{t}\oplus\mathrm{Span}(e_1, e_2, e_3)$ as maximal Killing algebra. For a fixed Ω_{12}, all these spaces have the same curvature tensor w.r.t. the frame (e_1, e_2, e_3). We are looking for a Lorentz Lie group structure on such a space. Supposing e'_1, e'_2, e'_3, with $e'_i=e_i+\alpha_i f$, generate a subalgebra of $\mathfrak{k}$. From (3.10) it follows that such a subalgebra exists iff $b^2-4\Omega_{12}>0$, and in this case α_1 is a solution of $\alpha^2-\alpha b+\Omega_{12}=0$ and $\alpha_2=\alpha_3$. Assume for simplicity that $\alpha_3=0$. The Lie subalgebra $\mathfrak{l}=\mathrm{Span}(e'_1, e'_2, e'_3)$ is solvable and centreless,

$$[e'_1, e'_2]=[e'_1, e'_3]=\alpha e'_1\,, \qquad [e'_2, e'_3]=b(e'_2-e'_3)\,,$$

and then the canonical form $\theta=\theta^i e'_i$ of L, the simply connected Lie group of Lie algebra $\mathfrak{l}$, satisfies the system

$$d\theta^1+\alpha\theta^1\wedge(\theta^2+\theta^3)=0\,, \qquad d\theta^2+b\theta^2\wedge\theta^3=0\,, \qquad d(\theta^2+\theta^3)=0\,.$$

L is the geometric realization of our $\mathfrak{h}\mathfrak{t}$-triple, and then $e'=(e'_1, e'_2, e'_3)$ is a field of orthoframes of L. A straightforward calculation shows that $(\nabla_{e'_2}R)(e'_1, e'_2, e'_1, e'_3)=-2b\Omega_{12}\neq 0$, showing that L is not a symmetric space.

Since $\mathfrak{l}$ is centreless it is isomorphic to its adjoint representation. Let

$$X_1=\mathrm{ad}\,e'_1\,,\ X_2=\mathrm{ad}\,\frac{1}{\alpha}(e'_2-e'_3),\ X_3=\mathrm{ad}\,\frac{1}{\alpha}e'_3\,,$$

and let $A=\exp(uX_1)\exp(vX_2)\exp(tX_3)$ be an arbitrary element in the Lie subgroup of $Gl(3,\mathbb{R})$ generated by ad $\mathfrak{l}$. Then the canonical form $\Theta=A^{-1}\,dA$ is easily seen to be

$$\exp(t)\,du\,X_1+\exp(bt/\alpha)\,dv\,X_2+dt\,X_3=\mathrm{ad}(\theta)\,,$$

and we identify $(\mathrm{Ad}(L), \|\theta\|_1^2)$ with $Sol(\mathrm{b},\alpha)$. □

Proposition 3.3. *Suppose $\beta\neq 0$ is the imaginary part of a complex root of the equation $z^2-bz+\Omega_{12}=0$. Then $D(b,\beta)=(\mathbb{R}^3, g)$ where g is given by*

$$g=\exp(bx)(\cos^2\beta x\cdot(dy)^2+2dx\cdot dt)-(dx)^2 \qquad (3.12)$$

is a degenerate l.h.pR. Any three-dimensional degenerate l.h.pR. is locally isometric to some $D(b,\beta)$.

Proof. Take an $\mathfrak{h}\mathfrak{t}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$, defined by (3.3)–(3.5) with $a=0<\Omega_{12}$, $b^2-4\Omega_{12}<0$. As in the proof of Proposition 3.2, $\mathfrak{k}=\mathbb{R}f\oplus\mathrm{Span}(e_1, e_2, e_3)$ is the maximal Killing algebra of the geometric realization of $(\mathfrak{p}, \Gamma, \bar{\Omega})$. The Lie algebra $\mathfrak{k}$ has no subalgebra transverse to $\mathfrak{h}\mathfrak{t}$, and $\mathfrak{h}\mathfrak{t}$ is degenerate w.r.t. the Killing form. To end the proof, it is enough to show that the geometric realization of this $\mathfrak{h}\mathfrak{t}$-triple is $D(b, \beta)$.

We claim that the geometric realization is diffeomorphic to $\mathbb{R}^3$.

Our $\mathfrak{h}\mathfrak{t}$-triple is closed, since $\mathfrak{k}$ is solvable. Let $\mathfrak{g}$ be the derived algebra of $\mathfrak{k}$; this is the three-dimensional nilpotent Lie algebra. Let K be the simply connected group of Lie algebra $\mathfrak{k}$, and let G be the subgroup of Lie algebra $\mathfrak{g}$ and $H = \exp(\mathbb{R}f)$.

Since G is a codimension 1 Lie subgroup of the solvable group K, K/G is $\mathbb{R}$, and since G is Nil, G/H is easily seen to be $\mathbb{R}^2$. The projections defining these quotients being trivial fibrations, let k: $\mathbb{R} \to K$, g: $\mathbb{R}^2 \to G$ be differentiable sections of these fibrations. Then the map $(a, b, c) \to k(a)g(b, c)H$ is a diffeomorphism from $\mathbb{R}^3$ to K/H, proving our claim.

Let $e_4 = f$ and let $\theta = \theta^i e_i$ be the canonical form of K. (3.10) yields

$$\begin{aligned}
&\mathrm{d}\theta^1 - (\theta^2 + \theta^3) \wedge \theta^4 = 0 , \\
&\mathrm{d}\theta^4 - (\Omega\theta^1 - b\theta^4) \wedge (\theta^2 + \theta^3) = 0 , \\
&\mathrm{d}\theta^2 + b\theta^2 \wedge \theta^3 + \theta^1 \wedge \theta^4 = 0 , \\
&\mathrm{d}\theta^3 - b\theta^2 \wedge \theta^3 - \theta^1 \wedge \theta^4 = 0 ,
\end{aligned} \tag{3.13}$$

with the global solution

$$\begin{aligned}
&\theta^2 = \exp(bx)(\beta u \cdot \mathrm{d}v + \mathrm{d}t) , \\
&\theta^2 + \theta^3 = \mathrm{d}x , \\
&(-\tfrac{1}{2}b + \mathrm{i}\beta)\theta^1 + \theta^4 = \beta \exp(\tfrac{1}{2}b + \mathrm{i}\beta)x \cdot (\mathrm{d}u + \mathrm{i}\,\mathrm{d}v) .
\end{aligned} \tag{3.14}$$

The Pfaffian system $\theta^1 = \theta^2 = \theta^3 = 0$ has the first integrals x, y, z, where

$$y = v + u \tan(\beta x) , \qquad z = -\tfrac{1}{2}\beta u^2 \tan(\beta X) + t . \tag{3.15}$$

Then, on K the projectable tensor

$$g = (\theta^1)^2 + (2\theta^2 - (\theta^2 + \theta^3))(\theta^2 + \theta^3)$$

is given by (3.12). □

Remarks. The Lorentz spaces in Propositions 3.2, 3.3, are locally the only possible three-dimensional nonsymmetric homogeneous spaces modelled on a symmetric space. Together with the lists in Refs. [6,] or [18], they give the full picture of the local metric structures of homogeneous Lorentz three-dimensional manifolds.

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References

[1] J.K. Beem and S.G. Harris, Nongeneric null vectors, Gen. Rel. Grav. 25(9) (1993) 963–975.
[2] J.K. Beem and P.E. Parker, Values of the pseudoriemannian sectional curvature, Comment. Math. Helv. 59 (1984) 319–331.
[3] D. Bini, P. Carini and R. Jentzen, Gravitoelectromagnetism: application to rotating Minkowski, Gödel and Kerr spacetimes, Sixth Marcel Grossman Meeting, Part A, B (Kyoto, 1991) (World Scientific, River Edge, NJ, 1992) pp. 1622–1624.
[4] E. Cartan, *Leçons sur la géométrie des espaces de Riemann* (Gauthier-Villars, 1946).
[5] M. Cahen, J. Leroy, M. Parker, F. Tricerri and L. Vanhecke, Lorentz manifolds modelled on a Lorentz symmetric space, J. Geom. Phys. 7 (1990) 571–591.
[6] L.A. Cordero and P.E. Parker, Left invariant Lorentzian metrics on 3 dimensional Lie groups, submitted for publication.
[7] P.M. Gadea and J.A. Oubina, Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures, Houston J. Math. 18 (3) (1992) 449–465.
[8] T. Koike, M. Tanimoto and A. Hosoya, Compact homogeneous universes, to appear in J. Math. Phys.
[9] O. Kowalski, Counter example to the second "Singer's Theorem", Ann. Global Anal. Geom. 8 (1990) 211–214.
[10] L. Nicolodi and F. Tricerri, On two theorems of I.M. Singer, Ann. Global Anal. Geom. 8 (1990) 193–209.
[11] M. Novelo, N.F. Santalo and M.E.X. Guimaraes, Sinchronized frames for Gödel universe, Gen. Rel. Grav. 19(2) (1987) 137–165.
[12] B. O'Neill, *Semi-Riemannian Geometry* (Academic Press, 1983).
[13] F.M. Paiva and C. Romero, The limits of Brans–Diecke spacetime: a coordinate free approach, Gen. Rel. Grav. 25(12) (1993) 1305–1318.
[14] R. Palais, A global formulation of the Lie theory of transformation Groups, Mem. Amer. Math. Soc. No. 22 (1957).
[15] V. Patrangenaru, Locally homogeneous Riemannian manifolds and Cartan triples, Geometria Dedicata 50 (1994) 143–164.
[16] V. Patrangenaru, Invariants of locally homogeneous pseudoRiemannian spaces, Preprint Series in Mathematics, INCREST, No. 6/1987.
[17] L.O. Piementel and J. Socorro, Bianchi VI_0 models in $N=2$, $D=2$ supergravity, Gen. Rel. Grav. 25(11) (1993) 1159–1161.
[18] S. Rahmani, Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension trois, J. Geom. Phys. 9 (1992) 295–302.
[19] M.P. Ryan and L.C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, 1975).
[20] N. Rosen, The energy of the universe, Gen. Rel. Grav. 26(3) (1994) 319–321.
[21] A. Sagle and R. Walde, *Introduction to Lie Groups and Lie Algebras* (Academic Press, 1973).
[22] F. Tricerri and L. Vanhecke, Variétés riemanniennes dont le tenseur de courbure est celui d'un espace symetrique irréductible, C.R. Acad. Sci. Paris Sér. I 302 (1986) 233–235.
[23] F. Tricerri, Locally homogeneous Riemannian manifolds, Rend. Sem. Mat. Univ. Pol. Torino 50 (4) (1993) 411–426.
[24] P.S. Wesson and L. Ponce de Leon, Kaluza–Klein theory and Machian cosmology, Gen. Rel. Grav. 26(6) (1994) 555–566.